Chapter 11

Reynolds-Stress and Related Models

11.1 Introduction

Reynolds-Stress Closure. In Reynolds-stress models, modelled transport equations are solved for the individual Reynolds stresses $\langle u_i u_j \rangle$ and for the dissipation ε (or for another quantity, e.g., ω, that provides a length or time scale of the turbulence). Consequently, the turbulent viscosity hypothesis is not needed; and so one of the major defects of the models described in the previous Chapter is eliminated.

The exact transport equation for the Reynolds stresses is obtained from the Navier-Stokes equations in Exercises 7.23–7.25 on pages 327–327: it is

$$\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t}\langle u_i u_j \rangle + \frac{\partial}{\partial x_k} T_{kij} = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \varepsilon_{ij}. \tag{11.1}$$

In a Reynolds-stress model, the "knowns" are $\langle \mathbf{U} \rangle$, $\langle p \rangle$, $\langle u_i u_j \rangle$ and ε. Thus in Eq. (11.1) both the mean-flow convection, $\bar{\mathrm{D}}\langle u_i u_j \rangle / \bar{\mathrm{D}}t$, and the production tensor, $\mathcal{P}_{ij}$ (Eq. 7.179), are in closed form. But models are required for the dissipation tensor ε_{ij} (Eq. 7.181), the pressure–rate-of-strain tensor $\mathcal{R}_{ij}$ (Eq. 7.187), and the Reynolds-stress flux T_{kij} (Eq. 7.195).

Outline of the Chapter. By far the most important quantity to be modelled is the pressure–rate-of-strain, $\mathcal{R}_{ij}$. This term is considered extensively in the next four Sections in the context of homogeneous turbulence. This includes (in Section 11.4) a description of Rapid Distortion Theory (RDT), which applies to a limiting case, and which provides useful insights. The extension to inhomogeneous flows is described in Section 11.6, and special near-wall treatments—both for k-ε and Reynolds-stress models—are described in Section 11.7. There is a vast literature on Reynolds-stress models, with many different proposals and variants: the emphasis here is on the fundamental concepts and approaches.

In Reynolds-stress models, $\mathcal{R}_{ij}$ is modelled as a local function of $\langle u_i u_j \rangle$, ε and $\partial \langle U_i \rangle / \partial x_j$. Elliptic relaxation models (Section 11.8) provide a higher level of closure and thereby allow the model for $\mathcal{R}_{ij}$ to be non-local.

Algebraic stress models and nonlinear turbulent viscosity models are described in Section 11.9. These are simpler models that can be derived from Reynolds-stress closures. The Chapter concludes with an appraisal of the relative merits of the range of models described.

Dissipation. The dissipation is treated summarily here. For high Reynolds number flows, a consequence of local isotropy is[1]

$$\varepsilon_{ij} = \tfrac{2}{3} \varepsilon \delta_{ij}. \tag{11.2}$$

This is taken as the model for ε_{ij}. For moderate Reynolds-number flows, this isotropic relation may not be completely accurate (see, e.g., Fig. 7.39 on page 326). But to an extent this is of no consequence because the anisotropic component (i.e., $\varepsilon_{ij} - \frac{2}{3}\varepsilon\delta_{ij}$) has the same mathematical properties as $\mathcal{R}_{ij}$, and so can be absorbed into the model for $\mathcal{R}_{ij}$. As discussed in Section 11.7, close to walls the dissipation is anisotropic, and different models are appropriate.

Reynolds Number. Most Reynolds-stress models contain no Reynolds-number dependence (except for near-wall treatments), and therefore they implicitly assume that the terms being modelled are independent of Reynolds number. For simplicity of exposition, we follow this expedient assumption. But it is good to remember that, in moderate Reynolds number experiments, and especially in DNS, there can be (usually modest) Reynolds-number effects.

11.2 Pressure–Rate-of-Strain

The fluctuating pressure appears in the Reynolds-stress equation (Eq. 7.178) most directly as the velocity-pressure-gradient tensor

[1]In this context it is not necessary to distinguish between the dissipation ε and the pseudodissipation $\tilde{\varepsilon}$.

$$\Pi_{ij} \equiv -\frac{1}{\rho}\left\langle u_i \frac{\partial p'}{\partial x_j} + u_j \frac{\partial p'}{\partial x_i} \right\rangle. \tag{11.3}$$

This can be decomposed (see Exercise 7.24 on page 327) into the pressure-transport term $-\partial T_{kij}^{(p)}/\partial x_k$, Eq. (7.192), and the pressure–rate-of-strain $\mathcal{R}_{ij}$

$$\mathcal{R}_{ij} \equiv \left\langle \frac{p'}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\rangle. \tag{11.4}$$

The trace of $\mathcal{R}_{ij}$ is zero ($\mathcal{R}_{ii} = 2\langle p'\nabla\cdot\mathbf{u}/\rho\rangle = 0$), and consequently the term does not appear in the kinetic energy equation: it serves to *redistribute* energy among the Reynolds stresses.

As observed by Lumley (1975), the decomposition of Π_{ij} into a redistribution term and a transport term is not unique. For example, an alternative decomposition is

$$\Pi_{ij} = \mathcal{R}_{ij}^{(a)} - \frac{\partial}{\partial x_\ell}\left(\tfrac{2}{3}\delta_{ij}T_\ell^{(p)}\right), \tag{11.5}$$

with

$$\mathcal{R}_{ij}^{(a)} \equiv \Pi_{ij} - \tfrac{1}{3}\Pi_{\ell\ell}\delta_{ij}, \tag{11.6}$$

and

$$\mathbf{T}^{(p)} \equiv \langle \mathbf{u}p'\rangle/\rho. \tag{11.7}$$

The significance of $\mathbf{T}^{(p)}$ is that the source of kinetic energy due to pressure transport is

$$\frac{1}{2}\Pi_{ii} = -\nabla\cdot\mathbf{T}^{(p)}. \tag{11.8}$$

In homogeneous turbulence the pressure transport is zero, and all redistributive terms are equivalent (e.g., $\Pi_{ij} = \mathcal{R}_{ij} = \mathcal{R}_{ij}^{(a)}$). In examining such flows (in this and the next three sections) we focus on the pressure–rate-of-strain $\mathcal{R}_{ij}$. For inhomogeneous flows it is a matter of convenience which decomposition to use; and in this case, as discussed in Section 11.6, there are reasons to favor the decomposition in terms of $\mathcal{R}_{ij}^{(a)}$, Eq. (11.5).

Importance of Redistribution. It is worth recalling the behavior of Π_{ij} in the turbulent boundary layer. In the budget for $\langle u^2\rangle$ (Fig. 7.35 on page 324), Π_{11} removes energy at about twice the rate of ε_{11}. These two sinks are (approximately) balanced by the production $\mathcal{P}_{11}$. In the budgets for $\langle v^2\rangle$ and $\langle w^2\rangle$ (Figs. 7.36 and 7.37) there is no production, but Π_{22} and Π_{33} are sources that approximately balance ε_{22} and ε_{33}. Thus energy is redistributed from the largest normal stress (which has all of the energy production) to the smaller normal stresses (which have no production). In the shear stress budget (Fig. 7.38 on page 325), the production $\mathcal{P}_{12}$ is approximately balanced by $-\Pi_{12}$, with the dissipation being small in comparison.

Evidently, along with production and dissipation, redistribution is a dominant process in the balance of the Reynolds stresses. Consequently, its modelling is crucial, and the subject of extensive research.

Poisson Equation for p'. Some insight into the pressure–rate-of-strain can be gained by examining the Poisson equation for pressure (see Section 2.5). The Reynolds decomposition of this equation (performed in Exercise 11.1) leads to a Poisson equation for p' with two source terms:

$$\frac{1}{\rho}\nabla^2 p' = -2\frac{\partial \langle U_i \rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j - \langle u_i u_j \rangle). \tag{11.9}$$

Based on this equation, the fluctuating pressure field can be decomposed into three contributions:

$$p' = p^{(r)} + p^{(s)} + p^{(h)}. \tag{11.10}$$

The *rapid pressure* $p^{(r)}$ satisfies

$$\frac{1}{\rho}\nabla^2 p^{(r)} = -2\frac{\partial \langle U_i \rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i}, \tag{11.11}$$

the *slow pressure* $p^{(s)}$ satisfies

$$\frac{1}{\rho}\nabla^2 p^{(s)} = -\frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j - \langle u_i u_j \rangle), \tag{11.12}$$

and the *harmonic* contribution $p^{(h)}$satisfies Laplace's equation $\nabla^2 p^{(h)} = 0$. Boundary conditions are specified on $p^{(h)}$—dependent on those on $p^{(r)}$ and $p^{(s)}$—so that p' satisfies the required boundary conditions.

The rapid pressure is so called because (unlike $p^{(s)}$) it responds immediately to a change in the mean velocity gradients. Also, in the rapid-distortion limit (i.e., $\mathcal{S}k/\varepsilon \to \infty$), the rapid pressure field $p^{(r)}$ has a leading-order effect, whereas $p^{(s)}$ is negligible (Section 11.4).

Corresponding to $p^{(r)}$, $p^{(s)}$ and $p^{(h)}$, the pressure–rate-of-strain can also be decomposed into three contributions, $\mathcal{R}_{ij}^{(r)}$, $\mathcal{R}_{ij}^{(s)}$, $\mathcal{R}_{ij}^{(h)}$, with obvious definitions, e.g.,

$$\mathcal{R}_{ij}^{(r)} \equiv \left\langle \frac{p^{(r)}}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\rangle. \tag{11.13}$$

As shown in the seminal work of Chou (1945), the Green's function solution to the Poisson equation (Eq. 2.48) can be used to express the pressure–rate-of-strain in terms of two-point velocity correlations. For example, in homogeneous turbulence, one contribution to $\mathcal{R}_{ij}^{(r)}$ is

$$\left\langle \frac{p^{(r)}}{\rho} \frac{\partial u_i}{\partial x_j} \right\rangle = 2 \frac{\partial \langle U_k \rangle}{\partial x_\ell} M_{i\ell jk}, \tag{11.14}$$

where the fourth-order tensor $\mathbf{M}$ is given by an integral of the two-point velocity correlation $R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r}) \rangle$:

$$M_{i\ell jk} = -\frac{1}{4\pi} \int \frac{1}{|\mathbf{r}|} \frac{\partial^2 R_{i\ell}}{\partial r_j \partial r_k} \mathrm{d}\mathbf{r}, \tag{11.15}$$

(see Exercise 11.2).

The valuable conclusions from these considerations are that there are three qualitatively different contributions to $\mathcal{R}_{ij}$. The rapid pressure involves the mean velocity gradients, and in homogeneous turbulence $\mathcal{R}_{ij}^{(r)}$ is directly proportional to $\partial \langle U_k \rangle / \partial x_\ell$ (Eq. 11.14). The slow pressure–rate-of-strain $\mathcal{R}_{ij}^{(s)}$ can be expected to be significant in most circumstances (except rapid distortion); and indeed, in decaying homogeneous anisotropic turbulence, $\mathcal{R}_{ij}^{(s)}$ is the only one of the three contributions that is non-zero. The harmonic component $\mathcal{R}_{ij}^{(h)}$ is zero in homogeneous turbulence, and is important only near walls: it is discussed in Section 11.7.5.

Exercise 11.1 Show that the Poisson equation for pressure (Eq. 2.42) can alternatively be written

$$\frac{1}{\rho} \nabla^2 p = -\frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} = -\frac{\partial^2 U_i U_j}{\partial x_i \partial x_j}. \tag{11.16}$$

Hence show that the mean pressure satisfies

$$\frac{1}{\rho} \nabla^2 \langle p \rangle = -\frac{\partial \langle U_i \rangle}{\partial x_j} \frac{\partial \langle U_j \rangle}{\partial x_i} - \frac{\partial^2 \langle u_i u_j \rangle}{\partial x_i \partial x_j}, \tag{11.17}$$

and that the fluctuation pressure satisfies Eq. (11.9).

Exercise 11.2 Consider homogeneous turbulence (in which the mean velocity gradient $\partial \langle U_k \rangle / \partial x_\ell$ is uniform). From Eqs. (2.48) and (11.11), show that the correlation at $\mathbf{x}$ between the rapid pressure and a random field $\phi(\mathbf{x})$ is given by

$$\frac{1}{\rho}\langle p^{(r)}(\mathbf{x})\phi(\mathbf{x})\rangle = \frac{1}{2\pi}\frac{\partial\langle U_k\rangle}{\partial x_\ell}\iiint_{-\infty}^{\infty}\left\langle\frac{\partial u_\ell(\mathbf{y})}{\partial y_k}\phi(\mathbf{x})\right\rangle\frac{\mathrm{d}\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, \quad (11.18)$$

where integration is over all $\mathbf{y}$. Comment on the behavior of the two-point correlation necessary for the integral to converge. Show that a contribution to $\mathcal{R}_{ij}^{(r)}$ is

$$\left\langle\frac{p^{(r)}}{\rho}\frac{\partial u_i}{\partial x_j}\right\rangle = \frac{1}{2\pi}\frac{\partial\langle U_k\rangle}{\partial x_\ell}\iiint_{-\infty}^{\infty}\frac{\partial^2}{\partial x_j\partial y_k}\langle u_i(\mathbf{x})u_\ell(\mathbf{y})\rangle\frac{\mathrm{d}\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \quad (11.19)$$

With the separation vector defined by $\mathbf{r}\equiv\mathbf{y}-\mathbf{x}$, show that this equation can be rewritten in terms of the two-point velocity correlation $R_{ij}(\mathbf{r})\equiv\langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle$ as

$$\left\langle\frac{p^{(r)}}{\rho}\frac{\partial u_i}{\partial x_j}\right\rangle = -\frac{1}{2\pi}\frac{\partial\langle U_k\rangle}{\partial x_\ell}\iiint_{-\infty}^{\infty}\frac{1}{|\mathbf{r}|}\frac{\partial^2 R_{i\ell}}{\partial r_j\partial r_k}\,\mathrm{d}\mathbf{r}. \quad (11.20)$$

Hence verify Eq. (11.14).

11.3 Return-to-Isotropy Models

11.3.1 Rotta's Model

The simplest situation in which to examine the slow pressure–rate-of-strain is decaying homogeneous anisotropic turbulence. In this case, there is no production or transport, and $\mathcal{R}_{ij}^{(r)}$ and $\mathcal{R}_{ij}^{(h)}$ are zero, so that the exact Reynolds-stress equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u_iu_j\rangle = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}. \quad (11.21)$$

Since the trace of $\mathcal{R}_{ij}^{(s)}$ is zero, the term has no effect on the turbulent kinetic energy. Its effect is on the distribution of energy among the Reynolds stresses, which can be examined through the normalized anisotropy tensor

$$b_{ij}\equiv\frac{\langle u_iu_j\rangle}{\langle u_ku_k\rangle} - \tfrac{1}{3}\delta_{ij} = \frac{a_{ij}}{2k}. \quad (11.22)$$

Taking ε_{ij} to be isotropic (Eq. 11.2), the evolution equation for b_{ij} is

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} = \frac{\varepsilon}{k}\left(b_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\varepsilon}\right), \tag{11.23}$$

(see Exercise 11.3).

It is natural to suppose that the turbulence has a tendency to become less anisotropic as it decays, and indeed such a tendency to *return to isotropy* is evident in Fig. 10.2 on page 372. Based on this notion, Rotta (1951) proposed the model

$$\begin{aligned}\mathcal{R}_{ij}^{(s)} &= -C_R\frac{\varepsilon}{k}\left(\langle u_i u_j\rangle - \tfrac{2}{3}k\delta_{ij}\right) \\ &= -2C_R\varepsilon b_{ij},\end{aligned} \tag{11.24}$$

where C_R is the "Rotta constant." Substituting this into Eq. (11.23) yields

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} = -(C_R - 1)\frac{\varepsilon}{k}b_{ij}, \tag{11.25}$$

showing that Rotta's model corresponds to a *linear* return to isotropy. Evidently a value of C_R greater than unity is required.

Exercise 11.3 From the definition of b_{ij} (Eq. 11.22) and from the Reynolds-stress evolution equation (Eq. 11.21), show that in decaying turbulence the exact equation for b_{ij} is

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} = \frac{\varepsilon}{k}\left(b_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\varepsilon} + \frac{1}{3}\delta_{ij} - \frac{\varepsilon_{ij}}{2\varepsilon}\right). \tag{11.26}$$

Hence show that Eq. (11.23) follows from the assumed isotropy of ε_{ij}.

Show that if instead ε_{ij} is taken to be proportional to $\langle u_i u_j\rangle$, then the resulting equation for b_{ij} is

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} = \frac{\mathcal{R}_{ij}^{(s)}}{2k}. \tag{11.27}$$

Show that if Rotta's model is used in this equation, the result is the same as Eq. (11.25) but with C_R in place of $(C_R - 1)$.
